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A generalization of power series to real several dimensions

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Introduction. In this paper I would like to propose a generalization of power series to real several dimensions.

Weierstrass laid the foundation of the theory of analytic functions on power series and analytic continuations. Elementary functions like e^z, $\sin z$, $\cos z$, and $\log(1+z)$ are all represented by convergent power series

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

with real coefficients a_n. These real analytic functions are however much better understood when we consider z as a complex variable rather than a real variable. One natural thought might be to consider z in the power series as a quaternion variable. This gives us a generalization of power series to real four dimension. However this simpleminded generalization is not satisfactory for a couple of reasons. First it gives a generalization only to four dimension. More importantly an essential difficulty arises when one tries analytic continuations; a translation of the quaternion variable in a power series results in a series which is not a power series any more since the quaternions are not commutative.

This difficulty of analytic continuations in a noncommutative situation lead me to consider the use of a special kind of Jordan algebras which I denote by J_n. These Jordan algebras J_n are explained in the first section.

In the second section I will give a short account of the Clifford algebras Cl_n and state the relationship between the Jordan algebra J_n and the Clifford algebra Cl_n. This relationship is in fact the reason why I started studying the Jordan algebras J_n. The generalization of power series is actually formulated in terms of the Jordan algebras J_n only. In this sense one can go without referring to the Clifford algebras at all. However without the perspective of the Clifford algebras the understanding of the generalization would be one-sided.

A theory of nonassociative polynomials over the Jordan algebra J_n will be developed in the third section, and the generalization of power series will be given in the forth section. The main features of the generalization given in this paper are the following:

(1) Every complex power series canonically gives rise to a local analytic transformation of the euclidean space of any dimension $n>2$ which extends the complex analytic function defined by the power series on the embedded complex plane.
(2) Analytic continuations can be performed.

One of my original motivations for this work was to try to define a class of analytic transformations of the n-dimensional euclidean space which is larger than the class of conformal maps but is not as large as the class of all analytic maps. From this point of view Theorem in the last section is a negative statement. It suggests that such a class cannot be defined directly by the generalization of power series given here.

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1. The Jordan algebra J_n. Most, if not all, of the materials presented in this and the next sections are well-known to algebraists in more general forms. The purposes of the first two sections are to provide notations suitable for the generalization of power series given later in this paper and to make the subjects more accessible to the readers in other fields of mathematics.

A real vector space J equipped with a bilinear map $\mu: J \times J \to J$, which will be called the multiplication, is a *Jordan algebra* if the following two conditions are satisfied:

$$\{a,b\} = \{b,a\}, \qquad (\forall a,b \in J), \tag{1.1}$$
$$\{\{a,a\},\{a,b\}\} = \{a,\{\{a,a\},b\}\}, \qquad (\forall a,b \in J). \tag{1.2}$$

Here $\{a,b\}$ denotes the image of $(a,b) \in J \times J$ under the multiplication μ. The notion of Jordan algebras was first introduced by P. Jordan in order to develop an algebraic theory of Hermitian operators (see [4]), and has later been used to construct certain Lie algebras. For a systematic treatment of Jordan algebras the reader is referred to [**3**].

The multiplication by an element $a \in J$ defines a linear transformation $T_a: J \to J$. Namely

$$T_a(x) = \{a,x\}. \tag{1.3}$$

Using this notation the condition (1.2) can be written as $T_a \circ T_{\{a,a\}} = T_{\{a,a\}} \circ T_a$. Notice that the multiplication is not assumed to be associative. Therefore the equality $T_a \circ T_b = T_{\{a,b\}}$ does not hold in general.

The Jordan algebra J_n is defined as follows. The underlying space of J_n is simply $\mathbb{R}^{n+1}$ which we regard as the direct sum $J_n^{(0)} \oplus J_n^{(1)}$ of the set of real numbers $J_n^{(0)} = \mathbb{R}$ and the n-dimensional euclidean space $J_n^{(1)} = \mathbb{R}^n$. Every element a of J_n can be uniquely written as $a = a^{(0)} + a^{(1)}$, where $a^{(0)} \in J_n^{(0)}$ and $a^{(1)} \in J_n^{(1)}$. We call the real number $a^{(0)}$ the real part of a, and the vector $a^{(1)}$ the imaginary part of a. For $a = a^{(0)} + a^{(1)} \in J_n$ and $b = b^{(0)} + b^{(1)} \in J_n$, the multiplication of J_n is defined by

$$\{a,b\} = (a^{(0)}b^{(0)} - a^{(1)} \cdot b^{(1)}) + (a^{(0)}b^{(1)} + b^{(0)}a^{(1)}), \tag{1.4}$$

where $a^{(1)} \cdot b^{(1)}$ denotes the inner product of the two vectors $a^{(1)}$ and $b^{(1)}$ in the euclidean space $J_n^{(1)}$. One can immediately see that the multiplication satisfies the commutativity condition (1.1). The condition (1.2) can also be verified directly by an easy computation, but the relationship between J_n and the Clifford algebra Cl_n explained in the next section will make it obvious. In any case J_n is therefore a Jordan algebra. Note that the multiplication by a real number $r \in J_n^{(0)}$ in the Jordan algebra J_n is simply the scalar multiplication by r. In particular the real number 1 acts as the unit element of J_n.

One would probably notice that the multiplication of J_n is exactly the multiplication of the complex numbers when $n = 1$. In fact we can canonically identify J_0 with the field of real numbers $\mathbb{R}$, and J_1 with the field of complex numbers $\mathbb{C}$. The multiplication of J_n is therefore associative for $n = 0$ and 1. However for $n > 1$ the Jordan algebra J_n is not associative.

The Jordan algebra J_n is naturally equipped with a norm as the $(n+1)$-dimensional euclidean space. Namely the norm of an element $a = a^{(0)} + a^{(1)} \in J_n$ is given by

$$|a| = \sqrt{a^{(0)2} + |a^{(1)}|^2}. \tag{1.5}$$

The following inequality is crucial in our generalization of power series.

PROPOSITION 1. $|\{a, b\}| \leq |a|\,|b|$ *for all* $a, b \in J_n$.

PROOF: A straightforward computation by the definition (1.5) shows

$$\begin{aligned}
&|a|^2|b|^2 - |\{a,b\}|^2 \\
&= (a^{(0)2} + |a^{(1)}|^2)(b^{(0)2} + |b^{(1)}|^2) - (a^{(0)}b^{(0)} - a^{(1)}\cdot b^{(1)})^2 - |a^{(0)}b^{(1)} + b^{(0)}a^{(1)}|^2 \\
&= |a^{(1)}|^2|b^{(1)}|^2 - |a^{(1)}\cdot b^{(1)}|^2 \\
&\geq 0,
\end{aligned}$$

where the last is the Schwarz's inequality.

We define the conjugate of an element a of the Jordan algebra $J_n$to be $\overline{a} = a^{(0)} - a^{(1)}$. This is a direct generalization of the conjugation for complex numbers. We have $\{a, \overline{a}\} = |a|^2 \in \mathbb{R}$ for any $a \in J_n$.

Let us investigate the linear transformation $T_a \colon J_n \to J_n$ defined for $a \in J_n$ by (1.3). First assume that $|a| = 1$. Then the element a is of the form

$$a = \cos\theta + (\sin\theta)v, \qquad (0 \leq \theta \leq \pi,\ v \in J_n^{(1)},\ |v| = 1).$$

Every element $z \in J_n$ can be written as

$$z = z_0 + z_1 v + w, \qquad (z_0, z_1 \in \mathbb{R},\ w \in J_n^{(1)},\ w \perp v).$$

Then by an easy computation we obtain

$$T_a(z) = (z_0\cos\theta - z_1\sin\theta) + (z_0\sin\theta + z_1\cos\theta)v + (\cos\theta)w.$$

Geometrically this means the following. Let P denote the 2-dimensional subspace of J_n spanned by 1 and v, and $P^\perp$ its orthogonal complement so that $J_n = P \oplus P^\perp$. Then the linear transformation $T_a \colon J_n \to J_n$ is the direct sum of the rotation of angle θ in P and the scalar multiplication by $\cos\theta$ in $P^\perp$. In general every element of J_n can be written as ra for some nonnegative real number r and for some $a \in J_n$ with $|a| = 1$. The linear transformation T_{ra} is the composition of T_a and the scalar multiplication by r.

As a consequence of the above argument we obtain:

PROPOSITION 2. *The linear transformation* $T_a \colon J_n \to J_n$ *is*

(1) *non-singular if the real part* $a^{(0)}$ *of* $a \in J_n$ *is nonzero,*

(2) *of rank at most two if* $a \in J_n^{(1)}$.

2. The Clifford algebra Cl_n. The fields of real and complex numbers are both associative and commutative. In order to extend these fields to higher dimensional real algebras we must give up one of the two properties. Giving up the associativity naturally leads to the Jordan algebras J_n explained in the previous section. If we abandon the commutativity instead we arrive at the concept of Clifford algebras.

In [**1**] Ahlfors initiated the use of Clifford algebras in the study of higher dimensional Möbius transformations, and I partially demonstrated its usefulness in [**6**] and [**7**]. The general philosophy behind these works is to simplify computations involved in the n-dimensional transformations by the use of an algebraic operation. And this is in fact the origin of the current work.

The Clifford algebras have mainly been studied for the sake of the spin groups and their representations. The readers interested in this aspect of Clifford algebras are referred to [**2**] and [**5**].

Consider the n-dimensional euclidean space $\mathbb{R}^n$ and the quadratic form $q: \mathbb{R}^n \to \mathbb{R}$ defined by $q(v) = -|v|^2$ for $v \in \mathbb{R}^n$, where $|v|$ is the norm of the vector v in the euclidean space $\mathbb{R}^n$. We denote by Cl_n the Clifford algebra associated to $(\mathbb{R}^n, q)$. In terms of an orthonormal basis $\{i_1, \ldots, i_n\}$ for $\mathbb{R}^n$, the Clifford algebra Cl_n can be described as follows. The Clifford algebra Cl_n is the real associative algebra generated by the elements $i_1, \ldots, i_n$ satisfying the following fundamental relations.

$$\begin{aligned} i_j^2 &= -1, \qquad (j = 1, \ldots, n), \\ i_j i_k &= -i_k i_j, \qquad (j, k = 1, \ldots, n,\ j \neq k). \end{aligned}$$

For $n = 0$, 1, and 2, the Clifford algebra Cl_n can be naturally identified with the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, and the skew field $\mathbb{H}$ of quaternions respectively.

Every element a of the Clifford algebra Cl_n can be uniquely written as

$$a = \sum a_I I, \qquad (a_I \in \mathbb{R}),$$

where the sum is taken over all the products I of generators of the form

$$I = i_{j_1} \cdots i_{j_r}, \qquad (1 \leq j_1 < \cdots < j_r \leq n).$$

In particular the Clifford algebra Cl_n is 2^n-dimensional as a real vector space. The number r of generators in the product I is called the degree of I.

We denote by $Cl_n^{(r)}$ the subspace of Cl_n spanned by the products I of generators of degree r. Then we have the direct sum decomposition

$$Cl_n = Cl_n^{(0)} \oplus Cl_n^{(1)} \oplus \cdots \oplus Cl_n^{(n)}.$$

Accordingly every element a of the Clifford algebra Cl_n is written as

$$a = a^{(0)} + a^{(1)} + \cdots + a^{(n)}, \tag{2.1}$$

where $a^{(r)} \in Cl_n^{(r)}$, $(r = 0, \ldots, n)$. The subspace $Cl_n^{(0)}$ is in fact a subalgebra, and can be naturally identified with the field of real numbers $\mathbb{R}$. The subspace $Cl_n^{(1)}$ is nothing but the euclidean space $\mathbb{R}^n$. We can easily verify that

$$uv + vu = -2u \cdot v, \tag{2.2}$$

for any vectors $u, v \in Cl_n^{(1)}$, and in particular that $v^2 = -|v|^2 = q(v)$.

The norm of an element $a = \sum a_I I$ of Cl_n is defined to be

$$|a| = \sqrt{\sum a_I^2}.$$

With respect to this norm the multiplication in the Clifford algebra Cl_n satisfies the following inequality:

$$|ab| \leq |a|\,|b|, \qquad (\forall a, b \in Cl_n). \tag{2.3}$$

In the Clifford algebra Cl_n there are three standard involutions. The main involution $a \mapsto a'$ is the automorphism of Cl_n characterized by $v' = -v$ for $v \in Cl_n^{(1)}$. The reversion $a \mapsto a^*$ is the anti-automorphism of Cl_n which acts identically on $Cl_n^{(1)}$. These two involutions of Cl_n commute with each other; their composition is called the conjugation and denoted by $\overline{a} = a'^* = a^{*\prime}$ $(a \in Cl_n)$. In terms of the direct sum decomposition (2.1) we have

$$\begin{aligned} a' &= a^{(0)} - a^{(1)} + a^{(2)} - a^{(3)} + \cdots + (-1)^n a^{(n)}, \\ a^* &= a^{(0)} + a^{(1)} - a^{(2)} - a^{(3)} + \cdots + (-1)^{\frac{n(n-1)}{2}} a^{(n)}, \\ \overline{a} &= a^{(0)} - a^{(1)} - a^{(2)} + a^{(3)} + \cdots + (-1)^{\frac{n(n+1)}{2}} a^{(n)}. \end{aligned}$$

For any element $a \in Cl_n$ the product $a\overline{a}$ is real and equal to $|a|^2$ if $n \leq 3$. But $a\overline{a}$ is not necessarily a real number in general; the real part $(a\overline{a})^{(0)}$ is always equal to $|a|^2$.

Now let us consider the subspace $Cl_n^{(0)} \oplus Cl_n^{(1)}$ of the Clifford algebra Cl_n. This subspace is not a subalgebra of Cl_n if $n > 1$. However the subspace is closed under the operation $Cl_n \times Cl_n \to Cl_n$, $(a, b) \mapsto \{a, b\}$ defined by

$$\{a, b\} = \frac{1}{2}(ab + ba). \tag{2.4}$$

The vector space $Cl_n^{(0)} \oplus Cl_n^{(1)}$ equipped with this bilinear operation is essentially the same thing as the Jordan algebra J_n. In fact using (2.2) we see for $r, s \in \mathbb{R}$ and for $u, v \in Cl_n^{(1)}$ that

$$\begin{aligned} \{r + u, s + v\} &= \frac{1}{2}\big((r + u)(s + v) + (s + v)(r + u)\big) \\ &= (rs - u \cdot v) + (rv + su). \end{aligned}$$

Compare this with (1.4). It is therefore quite natural to identify $J_n^{(0)}$ with $Cl_n^{(0)}$ and $J_n^{(1)}$ with $Cl_n^{(1)}$, and regard the Jordan algebra J_n as a subspace of the Clifford algebra Cl_n. We can then think that the multiplication of the Jordan algebra $J_n = Cl_n^{(0)} \oplus Cl_n^{(1)}$ is defined by (2.4) in terms of the associative multiplication of the Clifford algebra Cl_n. Notice that the property (1.2) of the Jordan algebra J_n is an obvious consequence of this observation. The two notions of the norms coincide. Therefore Proposition 1 is in fact an immediate corollary of (2.3) and the triangle inequality. Also the conjugation in the Jordan algebra J_n is nothing but the conjugation in the Clifford algebra Cl_n restricted to J_n.

If two elements a and b of the Jordan algebra J_n commute with each other as elements of the Clifford algebra Cl_n then by (2.4) we have $\{a, b\} = ab$. It follows that the k-th power of an element $a \in J_n$ is the same whether we use the multiplication of the Jordan algebra J_n or the one of the Clifford algebra Cl_n. We denote the k-th power of an element $a \in J_n$ simply by a^k.

Finally let us consider some low-dimensional cases. For $n = 0$ and 1 we have $J_0 = Cl_0 = \mathbb{R}$ and $J_1 = Cl_1 = \mathbb{C}$ respectively. The Clifford algebra Cl_2 is isomorphic to the skew field $\mathbb{H}$ of quaternions by the identification of i_1 with i, i_2 with j, and $i_1 i_2$ with k. Therefore in terms of the quaternions the Jordan algebra J_2 may be thought of as the 3-dimensional vector space

$$J_2 = \left\{a = a_0 + a_1 i + a_2 j \in \mathbb{H} \;\middle|\; a_0, a_1, a_2 \in \mathbb{R}\right\}$$

equipped with the multiplication defined by (2.4).

The Jordan algebra J_3 is naturally identified with the 4-dimensional subspace $Cl_3^{(0)} \oplus Cl_3^{(1)}$ of Cl_3. Actually one can express the Jordan algebra J_3 also in terms of the quaternions. There is a homomorphism $\Phi\colon Cl_3 \to \mathbb{H}$ of real associative algebras determined by $\Phi(i_1) = i$, $\Phi(i_2) = j$, and $\Phi(i_3) = k$. This homomorphism Φ maps the subspace $Cl_3^{(0)} \oplus Cl_3^{(1)}$ of Cl_3 isomorphically as a vector space onto the quaternion algebra $\mathbb{H}$. The restriction of the linear map Φ to the subspace $Cl_3^{(0)} \oplus Cl_3^{(1)}$ preserves the Jordan multiplication defined by (2.4). Therefore we can identify the Jordan algebra J_3 with the set $\mathbb{H}$ of quaternions equipped with the multiplication defined by (2.4).

3. J_n-polynomials. A *J_n-algebra* is a commutative, but not necessarily associative, real algebra A containing the unit element 1, together with a homomorphism $\epsilon_A\colon J_n \to A$. By a homomorphism I mean a linear map preserving the multiplication and the unit element. Let A be a J_n-algebra. Then we can multiply an element $a \in A$ by an element $s \in J_n$; we define

$$\{s, a\} = \{a, s\} = \{\epsilon_A(s), a\}$$

for $s \in J_n$ and $a \in A$. In particular the multiplication by a real number r is the same as the scalar multiplication by r in A; i.e., $\{r, a\} = ra$ for $r \in \mathbb{R}$ and $a \in A$.

Let A and B be two J_n-algebras, and ϵ_A, ϵ_B their associated homomorphisms. A homomorphism $\varphi\colon A \to B$ is called a *J_n-homomorphism* if it commutes with the associated homomorphisms; i.e., $\varphi \circ \epsilon_A = \epsilon_B$. This condition can also be written as

$$\varphi(\{s, a\}) = \{s, \varphi(a)\}, \qquad (\forall s \in J_n, \forall a \in A).$$

The above definitions are just a nonassociative analog of the definitions of commutative algebras over a commutative ring with the unit element 1, and their homomorphisms.

Now let us imitate the construction of polynomials over a commutative ring, and define a nonassociative version of polynomials.

Let X be an arbitrary set; we are actually interested in the case where the set X consists of one element z. We denote by $J_n[X]$ the free J_n-algebra generated by X. Namely $J_n[X]$ is a J_n-algebra containing the set X which satisfies the following universal property:

(P) Every map $\varphi: X \to A$ of the set X to any J_n-algebra A uniquely extends to a J_n-homomorphism $\widetilde{\varphi}: J_n[X] \to A$.

Such a J_n-algebra is easily proved to be unique up to J_n-isomorphisms. An element of the free J_n-algebra $J_n[X]$ is called a J_n-polynomial.

The J_n-polynomials can also be defined in a constructive manner as follows:

(1) Every element of the Jordan algebra J_n is a J_n-polynomial.
(2) Every element of the set X is a J_n-polynomial.
(3) If f and g are J_n-polynomials then the product $\{f, g\}$ is also a J_n-polynomial.
(4) If f and g are J_n-polynomials then the sum $f + g$ is also a J_n-polynomial.

Only those explicitly shown to be J_n-polynomials by (1)–(4) above are called J_n-polynomials. The product $\{f, g\}$ in (3) and the sum $f + g$ in (4) are formal expressions, unless both f and g are elements of J_n in which case they are respectively the product and the sum of f and g in the Jordan algebra J_n. For any real number r and for any J_n-polynomial f, we identify the product $\{r, f\}$ with the scalar multiple rf. Also the product $\{f, g\}$ is formally considered to be symmetric and bilinear. Namely we identify $\{f, g\}$ with $\{g, f\}$, $\{rf, g\}$ with $r\{f, g\}$, and$\{f + g, h\}$ with $\{f, h\} + \{g, h\}$ for any J_n-polynomials f, g, h, and for any real number r.

The set $J_n[X]$ of J_n-polynomials constructed this way is in fact a free J_n-algebra. First of all $J_n[X]$ is a commutative, nonassociative, real algebra in an obvious fashion. The unit element $1 \in J_n$ considered as a J_n-polynomial is the unit element of $J_n[X]$, and the inclusion map $\epsilon: J_n \to J_n[X]$ given by (1) is obviously a homomorphism. Therefore $J_n[X]$ is a J_n-algebra. Let A be any J_n-algebra with the associated homomorphism $\epsilon_A: J_n \to A$, and $\varphi: X \to A$ an arbitrary map of the set X to the J_n-algebra A. We can extend the map φ to a J_n-homomorphism $\widetilde{\varphi}: J_n[X] \to A$ recursively by the following:

(1) If $s \in J_n$ then $\widetilde{\varphi}(s) = \epsilon_A(s)$.
(2) If $x \in X$ then $\widetilde{\varphi}(x) = \varphi(x)$.
(3) If f and g are J_n-polynomials then $\widetilde{\varphi}(\{f, g\}) = \{\widetilde{\varphi}(f), \widetilde{\varphi}(g)\}$.
(4) If f and g are J_n-polynomials then $\widetilde{\varphi}(f + g) = \widetilde{\varphi}(f) + \widetilde{\varphi}(g)$.

The identifications of J_n-polynomials we make cause no difficulty since the multiplication by a real number r is also the scalar multiplication by r in A, and the multiplication in A is symmetric and bilinear. The map $\widetilde{\varphi}: J_n[X] \to A$ thus obtained is linear and preserves the multiplication by (3) and (4). The map $\widetilde{\varphi}$ also satisfies $\widetilde{\varphi} \circ \epsilon = \epsilon_A$ by (1), therefore is a J_n-homomorphism. On the other hand any J_n-homomorphism which extends the map $\varphi: X \to A$ necessarily satisfies the above conditions (1)–(4). Therefore it must coincide with the J_n-homomorphism $\widetilde{\varphi}$. This shows the uniqueness of the J_n-homomorphism $\widetilde{\varphi}$.

A J_n-polynomial is called a J_n-monomial if it can be obtained from (1) and (2) by multiplication (3) only. Every J_n-polynomial can be "expanded" into a finite sum of J_n-monomials. The number of times elements of the set X appear in a J_n-monomial f is called the degree of f, and is denoted by $\deg f$. Formally the degree of a J_n-monomial f is recursively defined by (1) $\deg s = 0$ for $s \in J_n$ ($s \neq 0$), (2) $\deg x = 1$ for $x \in X$, and (3) $\deg \{f, g\} = \deg f + \deg g$. We do not define the degree of the J_n-monomial 0; namely $\deg 0$ is indefinite.

We note that if $n \leq m$ then J_n-algebras and J_n-homomorphisms can be naturally considered as J_m-algebras and J_m-homomorphisms respectively. And a J_n-polynomial can be canonically thought of as a J_m-polynomial.

Now let us restrict ourselves to the case where the set X consists of one element z. In this case the set $J_n[X]$ of J_n-polynomials is simply denoted by $J_n[z]$. Elements of $J_n[z]$ are called J_n-polynomials of z. We use a symbol like $f(z)$, as well as f, to denote a J_n-polynomial of z.

Let A be any J_n-algebra. Then a J_n-polynomial $f \in J_n[z]$ canonically determines a transformation of A which we denote by the same symbol $f: A \to A$. The image $f(a)$ of an element $a \in A$ by f is given by "substituting" the variable z by a. Formally $f(a)$ is defined as follows. Define the map $\varphi: \{z\} \to A$ by $\varphi(z) = a$. By the universal property (P) this map φ uniquely extend to a J_n-homomorphism $\widetilde{\varphi}: J_n[z] \to A$. Then we define $f(a) = \widetilde{\varphi}(f)$. If f is a J_n-monomial of degree d then we have $f(ra) = r^d f(a)$ for all $r \in \mathbb{R}$ and for all $a \in J_n$.

The composition of two J_n-polynomials f and g is defined similarly. Define the map $\psi: \{z\} \to J_n[z]$ by $\psi(z) = g$. Using the J_n-homomorphism extension $\widetilde{\psi}: J_n[z] \to J_n[z]$, we define the composition $f \circ g \in J_n[z]$ to be $\widetilde{\psi}(f)$. It can be easily verified that $f \circ g(a) = f(g(a))$ for all $a \in A$.

As a special case of the above every J_n-polynomial $f(z) \in J_n[z]$ gives rise to a map $f: J_n \to J_n$. It should be noted that different J_n-polynomials may give the same map. For instance the two J_1-monomials z and $\{\{z, i_1\}, -i_1\}$ both give the identity map of J_1 since the Jordan algebra J_1 is associative. However these J_1-monomials give different maps of J_n to itself for $n > 1$; while the J_n-monomial z gives the identity map of J_n the other J_n-monomial defines a linear transformation of J_n of rank at most two by Proposition 2. A more suggestive example is the following. The J_2-monomial $\{\{\{z, i_1\}, i_2\}, i_1\}$ as considered as a map of J_n to itself is constantly zero for all $n \geq 2$.

These examples make it clear that we have to distinguish between form and matter; the J_n-polynomials and the transformations represented by the J_n-polynomials. Our standpoint is to deal with the "outer shells" rather than the "guts."

4. J_n-series. As stated in the previous section every J_n-polynomial is a finite sum of J_n-monomials. In this section we consider an infinite sum

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \tag{4.1}$$

of J_n-monomials $f_k(z)$ ($k = 0, 1, \dots$). Such an infinite sum is called a J_n-series of z, and the J_n-monomial $f_k(z)$ is called the k-th term. The set of J_n-series of z is denoted by $J_n[[z]]$. Note that we make no assumptions about the degree of the terms $f_k(z)$. In fact we allow infinitely many terms of the same degree to appear in a J_n-series; this generality is necessary for analytic continuations discussed later.

In order to talk about convergence of a J_n-series we first define the norm $\|f\|$ of a J_n-monomial f of z by

(1) $\|a\| = |a|$ for $a \in J_n$,
(2) $\|z\| = 1$, and
(3) $\|\{f,g\}\| = \|f\|\,\|g\|$ for any J_n-monomials f and g.

Then we have:

PROPOSITION 3. *If f is a J_n-monomial of z then*

$$|f(a)| \leq \|f\|\,|a|^{\deg f} \tag{4.2}$$

for all $a \in J_n$.

PROOF: (1) If $f \equiv b \in J_n$ then $\|f\| = |b|$, and $|f(a)| = |b| = \|f\|\,|a|^0$. (2) If $f \equiv z$ then $\|f\| = 1$, and $|f(a)| = |a| = \|f\|\,|a|^1$. (3) Suppose that the J_n-monomial f is written as a product $\{g,h\}$ for some J_n-monomials g and h of z which satisfy the condition (4.2). Then by Proposition 1 we obtain

$$\begin{aligned}
|f(a)| &= |\{g(a), h(a)\}| \\
&\leq |g(a)|\,|h(a)| \\
&\leq \|g\|\,|a|^{\deg g}\|h\|\,|a|^{\deg h} \\
&= \|\{g,h\}\|\,|a|^{\deg g + \deg h} \\
&= \|f\|\,|a|^{\deg f}.
\end{aligned}$$

This completes the proof of Proposition 3.

We define *the radius of norm convergence* of the J_n-series $f = \sum_{k=0}^{\infty} f_k$ to be

$$R(f) = \sup\{\, r \in \mathbb{R}_+ \mid \sum_{k=0}^{\infty} \|f_k\|\, r^{\deg f_k} < +\infty \,\}.$$

The following is an immediate corollary of Proposition 3.

COROLLARY 4. *Suppose that the radius of norm convergence $R(f)$ of a J_n-series $f(z) = \sum_{k=0}^{\infty} f_k(z) \in J_n[[z]]$ is positive. Then the series $\sum_{k=0}^{\infty} f_k(a)$ converges absolutely for all $a \in J_n$ with $|a| < R(f)$.*

A J_n-series with positive radius of norm convergence is said to be *norm convergent.* The norm convergence is a strong notion of convergence. For instance the J_n-series

$$f(z) = \sum_{k=0}^{\infty} \{\{\{z, i_1\}, i_2\}, i_1\}$$

is easily seen to have the radius of norm convergence 0, hence is not norm convergent. However this series considered as a series of maps of J_n is convergent for all $z \in J_n$ in the strongest possible sense, since each term is constantly zero. This example also shows that the notion of norm convergence explicitly depends on the formal expression of a J_n-series rather than the transformation represented by the J_n-series.

Analytic continuations are formulated as follows. Let

$$f(z) = \sum_{k=0}^{\infty} f_k(z) \tag{4.3}$$

be a norm convergent J_n-series and $R(f)$ its radius of norm convergence. Suppose that a is a point in J_n such that $|a| < R(f)$. Now let $z = a + w$ in the J_n-series (4.3) and expand the terms $f_k(a+w)$ with respect to the variable w so that

$$f_k(a+w) = \sum_{l=1}^{m_k} f'_{kl}(w),$$

where each $f'_{kl}(w)$ is a J_n-monomial of w. Thus we obtain a new J_n-series

$$f'(w) = \sum_{k=0}^{\infty} \sum_{l=1}^{m_k} f'_{kl}(w). \tag{4.4}$$

PROPOSITION 5. *The radius of norm convergence of the J_n-series* (4.4) *is at least* $R - |a|$.

We first prove a lemma.

LEMMA 6. *Let $f(z)$ be a J_n-monomial of z and a an element of J_n. Suppose that the J_n-polynomial $f(a+w)$ of w is expanded as*

$$f(a+w) = \sum_{l=1}^{m} f'_l(w)$$

into the sum of J_n-monomials $f'_l(w)$ $(l = 1, \ldots, m)$. Then

$$\|f\| \, (|a| + r)^{\deg f} = \sum_{l=1}^{m} \|f'_l(w)\| \, r^{\deg f'_l} \tag{4.5}$$

for any positive real number r.

PROOF: (1) If f is a constant $b \in J_n$ then $m = 1$, $f'_1 = b$, and the formula (4.5) obviously holds. (2) If $f \equiv z$ then both sides of (4.5) are equal to $|a| + r$. (3) Suppose that the J_n-monomial f is written as a product $\{g, h\}$ of two J_n-monomials g and h for which the lemma has been proved. Namely assume that there are expansions

$$g(a+w) = \sum_{k=1}^{p} g'_k(w),$$

$$h(a+w) = \sum_{l=1}^{q} h'_l(w),$$

and that we have

$$\|g\|\,(|a|+r)^{\deg g} = \sum_{k=1}^{p} \|g'_k\|\, r^{\deg g'_k},$$

$$\|h\|\,(|a|+r)^{\deg h} = \sum_{l=1}^{q} \|h'_l\|\, r^{\deg h'_l}.$$

The expansion of $f(a+w) = \{g(a+w), h(a+w)\}$ into the sum of J_n-monomials of w is given by

$$f(a+w) = \sum_{k=1}^{p}\sum_{l=1}^{q} \{g'_k(w), h'_l(w)\}.$$

Then we have

$$\begin{aligned}
&\|f\|\,(|a|+r)^{\deg f} \\
&= \|\{g,h\}\|\,(|a|+r)^{\deg g+\deg h} \\
&= \|g\|\,(|a|+r)^{\deg g}\|h\|\,(|a|+r)^{\deg h} \\
&= \left(\sum_{k=1}^{p} \|g'_k\|\, r^{\deg g'_k}\right)\left(\sum_{l=1}^{q} \|h'_l\|\, r^{\deg h'_l}\right) \\
&= \sum_{k=1}^{p}\sum_{l=1}^{q} \|g'_k\|\,\|h'_l\|\, r^{\deg g'_k+\deg h'_l} \\
&= \sum_{k=1}^{p}\sum_{l=1}^{q} \|\{g'_k, h'_l\}\|\, r^{\deg \{g'_k,h'_l\}}.
\end{aligned}$$

This completes the proof of Lemma 6.

Now it is easy to prove Proposition 5. Let r be an arbitrary real number such that $|a|+r < R(f)$. By the definition of the radius of norm convergence of $f(z)$ we have

$$\sum_{k=0}^{\infty} \|f_k\|\,(|a|+r)^{\deg f_k} < +\infty.$$

By Lemma 6 this is equivalent to

$$\sum_{k=0}^{\infty}\sum_{l=1}^{m_k} \|f'_{kl}\|\, r^{\deg f'_{kl}} < +\infty.$$

Therefore the radius of norm convergence of the J_n-series (4.4) is at least r. Since r was an arbitrary real number such that $r < R(f) - |a|$ this means that the radius of norm convergence of the J_n-series (4.4) is at least $R(f) - |a|$. This completes the proof of Proposition 5.

Let us look at some examples. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{4.6}$$

be a convergent power series in the usual sense. First assume that the coefficients a_k are all real numbers. The standard power series expansions of functions like e^z, $\sin z$, $\cos z$, $\log(1+z)$ are all of this type. Such a power series is automatically a J_n-series for all $n = 0, 1, \ldots$ The radius of norm convergence $R(f)$ of the J_n-series is simply the radius of convergence of the power series (4.6) in the usual sense. Therefore the J_n-series canonically gives a family of maps

$$f_n \colon J_n \to J_n, \qquad (n = 0, 1, \ldots)$$

defined at least in the inside of the ball of radius $R(f)$. The map $f_0 \colon J_0 \to J_0$ is simply the real analytic function given by the power series (4.6), and the map $f_1 \colon J_1 \to J_1$ is its extension to the complex plane. For higher dimensions the map f_n is obtained by "rotating" the map f_1 around "the real axis."

Suppose next that the coefficients a_k of the power series (4.6) are complex numbers. While the power z^k naturally makes sense in J_n, multiplying z^k by anything other than a real number using the Clifford multiplication does not make a good sense in the Jordan algebra J_n. The natural thing to do here is to replace the usual multiplication of a_k and z^k by the Jordan multiplication $\{a_k, z^k\}$. Namely we consider the J_n-series

$$\sum_{k=0}^{\infty} \{a_k, z^k\} \tag{4.7}$$

as the generalization of the power series (4.6). Since the Jordan multiplications in J_0 and J_1 coincide with the usual multiplications of $\mathbb{R}$ and $\mathbb{C}$ respectively the J_n-series (4.7) is in fact a direct extension of (4.6).

5. Polynomial maps. As we saw in the sections 3 and 4 J_n-polynomials and norm convergent J_n-series are not in one to one correspondence with the transformations of the space J_n represented by them. Therefore the following problem is of fundamental importance in the study of the subject. Let n be a non-negative integer.

PROBLEM 1. *Characterize the J_n-polynomials (norm convergent J_n-series) which define the zero map of J_n to itself.*

Or more strongly:

PROBLEM 2. *Give the normal forms of J_n-polynomials (J_n-series).*

Problem 1 for J_n-polynomials can be given an answer as follows. Suppose that $f(z)$ is a J_n-polynomial. Consider z and w specifically as variables in J_n and let

$$w = f(z). \tag{5.1}$$

Write the variables z and w as

$$z = z_0 + z_1 i_1 + \cdots + z_n i_n,$$
$$w = w_0 + w_1 i_1 + \cdots + w_n i_n,$$

using the real variables $z_0, z_1, \ldots, z_n$ and $w_0, w_1, \ldots, w_n$. Then the relation (5.1) gives a family of polynomials $f_0, f_1, \ldots, f_n \in \mathbb{R}[z_0, z_1, \ldots, z_n]$ in the usual sense so that

$$w_j = f_j(z_0, z_1, \ldots, z_n), \qquad (j = 0, 1, \ldots, n).$$

Then the answer to Problem 1 is the following: The J_n-polynomial $f(z) \in J_n[z]$ defines the zero map $f: J_n \to J_n$ if and only if these polynomials $f_0, f_1, \ldots, f_n \in \mathbb{R}[z_0, z_1, \ldots, z_n]$ are all zero.

However this answer to Problem 1 is somehow against the basic idea of "simplifying computations involved in the n-dimensional transformations by the use of an algebraic operation." What we really want is a characterization which does not refer to the coordinates.

In any case the transformation $f: J_n \to J_n$ given by a J_n-polynomial $f(z) \in J_n[z]$ is after all a polynomial map in the usual sense. If $n \neq 1$ then the converse is also true.

PROPOSITION 7. *If $n \neq 1$ then every polynomial transformation $f: J_n \to J_n$ can be represented by a J_n-polynomial $f(z) \in J_n[z]$.*

PROOF: If $n = 0$ then the statement is obvious; let us assume that $n > 1$. Every element $z \in J_n$ can be uniquely written as

$$z = z_0 + z_1 i_1 + \cdots + z_n i_n, \qquad (z_0, z_1, \ldots, z_n \in \mathbb{R}).$$

Therefore the real part z_0 and the coefficients z_j of i_j $(j = 1, \ldots, n)$ can be considered as real valued functions on J_n. First we show that these real valued functions are representable by J_n-polynomials. Let us consider z_j $(j = 1, \ldots, n)$. We can take an index k different from j since $n > 1$. Then by an easy computation we get

$$\{z, i_j\} = -z_j + z_0 i_j,$$
$$\{\{z, i_j\}, i_k\} = -z_j i_k,$$

therefore

$$\{\{\{z, i_j\}, i_k\}, i_k\} = z_j.$$

Similarly we obtain

$$\{\{\{\{z, i_1\}, i_1\}, i_2\}, i_2\} = z_0.$$

Since the Jordan multiplication coincides with the usual multiplication of the real numbers, we can multiply these J_n-polynomials together with a real number to get a J_n-monomial which represents any monomial of $z_0, z_1, \ldots, z_n$ in the usual sense. We then add some of these J_n-monomials to get J_n-polynomials $f_0, f_1, \ldots, f_n \in J_n[z]$ which represent arbitrary polynomial functions. Finally let

$$f(z) = f_0 + \{f_1, i_1\} + \cdots + \{f_n, i_n\}.$$

This way we can find a J_n-polynomial which represents any desired polynomial map.

For $n = 1$ the Jordan algebra J_1 is the field of complex numbers and the transformations $f\colon J_1 \to J_1$ given by J_1-polynomials are just complex polynomial functions. Since such maps are always conformal, not all polynomial maps on J_1 are representable by J_1-polynomials. Contrary to the similarity of outside appearances, the structures of the J_n-polynomials are quite different for $n = 1$ and for $n > 1$.

Analytic maps are just the limits of polynomial maps. As a corollary of the above we have the following:

THEOREM. *If $n \neq 1$ then every real analytic map $f\colon J_n \to J_n$ defined on a neighborhood of the origin can be represented by a norm convergent J_n-series $f(z) \in J_n[[z]]$.*

PROOF: Let $f\colon J_n \to J_n$ be any real analytic map defined on a neighborhood of the origin. Write $w = f(z)$ in terms of the coordinates as

$$w_j = f_j(z_0, \dots, z_n), \qquad (j = 0, \dots, n).$$

Then expand each function f_j as a power series in $n+1$ variables $z_0, \dots, z_n$;

$$f_j(z_0, \dots, z_n) = \sum_{k_0=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} c_{j k_0 \cdots k_n} z_0^{k_0} \cdots z_n^{k_n},$$

where the coefficients $c_{j k_0 \cdots k_n}$ are real numbers. Since the power series is convergent in a neighborhood of $0 \in J_n$ there is a positive real number r such that

$$\sum_{k_0=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} |c_{j k_0 \cdots k_n}| \, r^{k_0 + \cdots + k_n} < +\infty, \qquad (j = 0, \dots, n). \tag{5.2}$$

As in the proof of Proposition 7 we can express each term $z_0^{k_0} \cdots z_n^{k_n}$ as a J_n-monomial $g_{k_0 \cdots k_n}(z)$ of degree $k_0 + \cdots + k_n$ satisfying $\|g_{k_0 \cdots k_n}\| = 1$. The J_n-series

$$f(z) = \sum_{k_0, \dots, k_n = 0}^{\infty} c_{0 k_0 \cdots k_n} g_{k_0 \cdots k_n} + \sum_{j=1}^{n} \sum_{k_0, \dots, k_n = 0}^{\infty} c_{j k_0 \cdots k_n} \{g_{k_0 \cdots k_n}, i_j\}$$

is norm convergent because

$$\begin{aligned}
&\sum_{k_0, \dots, k_n = 0}^{\infty} \|c_{0 k_0 \cdots k_n} g_{k_0 \cdots k_n}\| \, r^{\deg g_{k_0 \cdots k_n}} \\
&\qquad + \sum_{j=1}^{n} \sum_{k_0, \dots, k_n = 0}^{\infty} \|c_{j k_0 \cdots k_n} \{g_{k_0 \cdots k_n}, i_j\}\| \, r^{\deg \{g_{k_0 \cdots k_n}, i_j\}} \\
&= \sum_{k_0, \dots, k_n = 0}^{\infty} |c_{0 k_0 \cdots k_n}| \, r^{k_0 + \cdots + k_n} + \sum_{j=1}^{n} \sum_{k_0, \dots, k_n = 0}^{\infty} |c_{j k_0 \cdots k_n}| \, r^{k_0 + \cdots + k_n} \\
&< +\infty
\end{aligned}$$

by (5.2). The norm convergent J_n-series $f(z) \in J_n[[z]]$ thus obtained obviously represents the given real analytic map f. This completes the proof of Theorem.

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